Complexity and ultradiffusion

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Abstract. We present the exact solution to the problem of ultradiffusion in an arbitrary hierarchical space. We derive rigorous upper and lower bounds for the dynamic exponent describing the decay of the autocorrelation function. We show that the upper bound is saturated by both uniformly and randomly multifurcating hierarchical trees and identify a class of highly unbalanced trees that saturate the lower bound. We conclude that the speed of relaxation is a measure of the complexity or lack of self-similarity of the underlying tree. We point out that complexity may be revealed by the temperature dependence of the dynamic exponent and, in particular, by the nature of the transition from exponential to power-law decay.

1. Introduction

A large variety of natural and artificial systems have an exact or approximate hierarchical structure (Simon 1962, Rammal *et al* 1986). Examples range from the reporting schemes in social organisations, to the way macromolecules are built out of atomic constituents or organisms out of cells, to the way individual spins can be collected into larger blocks at the critical point of ferromagnets. More recently, it has been realised that a hierarchical organisation of states appears spontaneously in the low-temperature phase of the mean-field theory spin glass (Sherrington and Kirkpatrick 1975, Mézard *et al* 1984a, b) and it has been conjectured that the same holds true for glasses, hard combinatorial optimisation problems (Kirkpatrick and Toulouse 1985, Bachas 1985, Bouchaud and Le Doussal 1986, Sorkin *et al* 1986, Fu and Anderson 1986), the conformational substates of proteins (Ansari *et al* 1985, Stein 1985) and other complex frustrated systems.

A common feature of all these systems is that, because of the existence of many different timescales, they appear to relax slower than exponentially when perturbed (Kohlrausch 1847, Williams and Watts 1970, Palmer *et al* 1984). To model this behaviour several authors have considered diffusive processes on hierarchical structures. The earliest variant proposed by Huberman and Kerzberg (1985), and further analysed by Teitel and Domany (1985a, b) and Maritan and Stella (1986a, b), consisted of a particle diffusing over a hierarchical array of energy barriers in one dimension. Allowing for long-range hoppings leads to the problem of diffusion in a truly ultrametric space, which has been analysed by a number of authors (Schreckenberg 1985, Ogielski and Stein 1985, Paladin *et al* 1985, Zheng 1986, Blumen *et al* 1986). Finally, a third related process is the random walk on the backbone of a tree, which has been used

to model the observed stickiness in the chaotic transport of particles (Meiss and Ott 1985, 1986, Grossman *et al* 1985).

A major limitation of all these treatments is that they only apply to the simple case of uniform hierarchical structures, described by trees whose branches at any given hierarchy level are totally indistinguishable. The intrinsic self-similarity of such trees allows for the application of renormalisation group techniques, so that ultradiffusion is in this case only a variant of the extensively studied diffusion on fractals (de Gennes 1976, Alexander and Orbach 1982, Rammal and Toulouse 1983). Hierarchical structures need not, however, be self-similar. For instance, it is hard to imagine how the metastable states of a spin glass could be represented by the indistinguishable leaves of a uniform tree, when different free energy valleys are known to carry different weights (Mézard *et al* 1985). And in a more general context, it is precisely the lack of self-similarity, or the appearance of different interactions at every new level of the hierarchy, that accounts for the complexity of hierarchical systems such as biological organisms and social organisations. It is therefore interesting to determine how hierarchical dynamics may depend on the underlying tree structure. This is the subject of this paper, a summary of which has already been published elsewhere (Bachas and Huberman 1986).

We will restrict ourselves here to the problem of diffusion in a truly ultrametric space. In § 2 we will solve this problem exactly, without making any assumptions concerning the structure of the underlying tree. We will obtain, in particular, a closed expression for the average autocorrelation function, whose asymptotic decay describes the rate of relaxation of the system.

A central thesis of this paper is that the dynamic exponent ν that characterises this asymptotic decay has the qualitative features of a measure of physical complexity introduced by Huberman and Hogg (1986), namely it is sensitive to the absence of self-similarity, rather than the randomness, or detailed information content of the underlying tree. To be precise, we derive a rigorous upper bound for ν, and show, in §§ 3 and 4, that it is saturated by both uniformly and randomly multifurcating trees. Such trees are therefore optimal in that they lead to fastest relaxation. In § 5 we also derive a lower bound for ν, and identify a class of non-self-similar unbalanced trees that saturate it. All these results have recently been shown to hold also for the critical percolation threshold by Bachas and Wolff (1987), which can be related to the complexity of games.

A corollary of our results is that in thermally activated processes the temperature dependence of the dynamic exponent is not universal, but varies with the underlying tree structure. In particular, the transition to instability is continuous for self-similar trees, and discontinuous for unbalanced ones. This is explained in § 6 which also contains some concluding remarks.

2. The general solution

Ultradiffusion is described by the dynamical equation

$$\frac{dP_i}{dt} = \sum_{j=1}^{N} \varepsilon_{ij} P_j \tag{2.1}$$

where $i = 1, \ldots, N$ labels the states of the system, $P_i(t)$ is the probability of finding the system at state i at time t, and the symmetric transition matrix ε satisfies the

ultrametric property

$$\varepsilon_{ij} \geq \min(\varepsilon_{ik}, \varepsilon_{jk}) \tag{2.2}$$

for any three distinct states i, j and k. If ε represents hopping rates between sites i and j, equation (2.2) implies that the two smallest hopping rates out of a triplet are equal. The diagonal transition elements are fixed by the requirement that probability be conserved, i.e.

$$\varepsilon_{ii} = -\sum_{j \neq i} \varepsilon_{ij}. \tag{2.3}$$

The ultrametric property (2.2) is equivalent to saying that the states of the system can be represented as the leaves of a generic tree, where the hopping rate between any two of them, say i and j, is only a function of the nearest common ancestor A of i and j on the tree,

$$\varepsilon_{ij} = \varepsilon_{ji} = \varepsilon_{A(ij)}$$

with ε_A decreasing monotonically as one climbs along any path towards the root. By appropriately stretching the tree, we may always assume without loss of generality that $\varepsilon_A = \exp(-h_A)$, where h_A is the height of the branching point A from the bottom of the tree, as shown in figure 1. We shall refer to such a tree as a 'metric tree' to stress the fact that both its topology and the heights of its branches matter. There is one hierarchical transition matrix and one ultradiffusion problem for every metric tree.
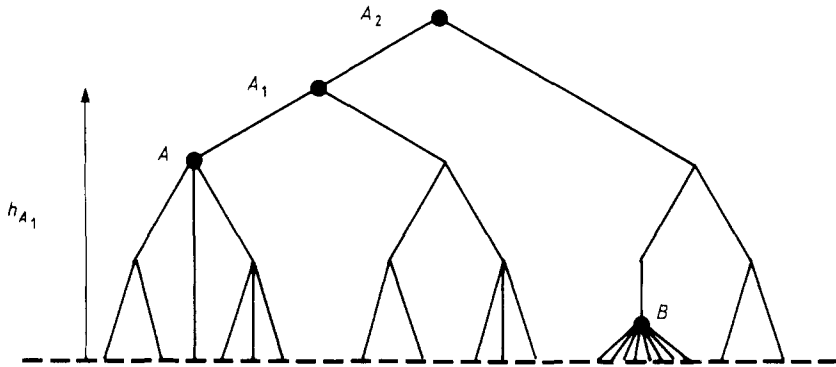

Figure 1. A generic tree illustrating our notation. The node A has three sons ($S_A = 3$) and six final descendants ($N_A = 6$). A_1 is his father and A_2 the grandfather, that also happens to be the root of the tree. The height of a node as measured from the bottom leaves is minus the logarithm of the corresponding transition rate. The node B illustrates how the weight of a state can be effectively increased (by a factor eight in the figure) by letting it multifurcate at low altitude.

An important remark is in order here: although the assumption of symmetric transition probabilities may seem overly restrictive, this is not the case since we can effectively increase the weight of any state by letting it multifurcate appropriately at low altitude as shown in figure 1.

Before proceeding to an exact solution of equation (2.1), let us first introduce some useful notation and terminology. We shall denote by B_n the unique nth ancestor of any branch point or tree leaf B ($B_0 = B$ by convention, B_1 is the father, B_2 the grandfather, and so on, up to the patriarch or root). N_B will stand for the total number

of final descendants or tree leaves generated by B ($N_B = 1$ if B is itself a leaf), and S_B for the number of sons, or immediate offspring. We also introduce the characteristic function

$$X_i(B) = \begin{cases} 1 & \text{if } i \text{ is a descendant of } B \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1, \ldots, N$ runs over the leaves of the tree. For convenience we will assume that branchings may only occur at integral multiples of some minimum adjustable height interval Δh, and will occasionally refer to all branches at height $h = m\Delta h$ as the mth generation. If $n(h)$ is their total number, we define the silhouette slopes, which measure the rate of population growth at height h, by

$$s(h) = -\frac{\Delta \log n(h)}{\Delta h} = \frac{1}{\Delta h} \log\left(\frac{n(h)}{n(h + \Delta h)}\right). \tag{2.4a}$$

We shall refer to its average asymptotic value

$$s \equiv \lim_{h' \gg h \to \infty} \int_{h/(h'-h)}^{h'} S \tag{2.4b}$$

simply as the tree's silhouette. Large and small values of silhouettes correspond to asymptotically fat and thin trees, respectively.

We are now ready to obtain the complete set of eigenvectors and eigenvalues of any hierarchical transition matrix ε. To this end, consider first the action of ε on the characteristic function of some branch point or tree leaf B:

$$\sum_{j=1}^{N} \varepsilon_{ij} X_j(B) = -\sum_{j=1}^{N} \varepsilon_{ij}(1 - X_j(B))$$

$$= \begin{cases} N_B \exp(-h_{A(i,B)}) & \text{if } X_i(B) = 0 \\ -\sum_{n=1}^{\text{root}} (N_{B_n} - N_{B_{n-1}}) \exp(-h_{B_n}) & \text{if } X_i(B) = 1 \end{cases} \tag{2.5}$$

where $A(i, B)$ is the nearest common ancestor of i and B, and by slight abuse of notation $\Sigma_{n=1}^{\text{root}}$ stands for a summation over all of the ancestors of B, up to and including the root.

We next define the vector

$$V_i(B, \tilde{B}) = \frac{1}{N_B} X_i(B) - \frac{1}{N_{\tilde{B}}} X_i(\tilde{B}) \tag{2.6}$$

for any two brothers B and $\tilde{B}$. Since $B_n = \tilde{B}_n$ for all $n > 0$, and $A(i, B) = A(i, \tilde{B})$ for any i that is neither a descendant of B nor of $\tilde{B}$, we easily deduce from equation (2.5) that $V(B, \tilde{B})$ is an eigenvector of ε, with eigenvalue λ that only depends on the common father of B and $\tilde{B}$, i.e.

$$\lambda(B_1) \equiv -\frac{1}{\tau_{B_1}} = -N_{B_1} \exp(-h_{B_1}) - \sum_{n=2}^{\text{root}} (N_{B_n} - N_{B_{n-1}}) \exp(-h_{B_n}). \tag{2.7}$$

Consequently, there are $\frac{1}{2} S_{B_1}(S_{B_1} - 1)$ degenerate eigenvectors of type (2.6), corresponding to all pairs of sons of B_1. A convenient basis for the subspace they span consists of the following vectors, one for each son of B_1:

$$V_i(B) \equiv \sum_{\text{brothers } \tilde{B} \text{ of } B} \frac{N_B}{N_{B_1}} V_i(B, \tilde{B}) = \frac{1}{N_B} X_i(B) - \frac{1}{N_{B_1}} X_i(B_1). \tag{2.8}$$

These satisfy the linear relationship

$$\sum_{\text{brothers} B} N_B V(B) = 0$$

so that only $(S_{B_1} - 1)$ of them are linearly independent. Since

$$\sum_{\text{branchpoints} B} (S_B - 1) = N - 1$$

there is a single missing eigenvector of the $N \times N$ transition matrix ε. This corresponds to the steady state of equal probability for all sites, which we denote by

$$V_i(\text{root}) = 1/N \qquad \sum_{j=1}^{N} \varepsilon_{ij} V_j(\text{root}) = 0. \tag{2.9}$$

Going back to equation (2.1), consider a particle that starts out at time zero at a given tree leaf L. Writing the initial condition as

$$P_i(t = 0) = \delta_{L,i} = \sum_{n=0}^{\text{root}} V_i(L_n)$$

we immediately deduce that at later times

$$P_i(t) = \sum_{n=1}^{\text{root}} V_i(L_{n-1}) \exp(-t/\tau_{L_n}) + \frac{1}{N}. \tag{2.10}$$

From this we can exactly calculate any quantity of interest. We will concentrate on the autocorrelation function, i.e. the probability that the particle returns to its point of departure; this is

$$P_L(t) = \frac{1}{N} + \sum_{n=1}^{\text{root}} \left(\frac{1}{N_{L_{n-1}}} - \frac{1}{N_{L_n}} \right) \exp(-t/\tau_{L_n}). \tag{2.11}$$

We will be mainly interested in this quantity averaged over all initial conditions L, which can be written

$$P(t) = \frac{1}{N} + \sum_{\text{branchpoints} B} \frac{1}{N} (S_{B-1}) \exp(-t/\tau_B) \tag{2.12}$$

where, for later ease of reference, we recall that

$$\frac{1}{\tau_B} = N_B \exp(-h_B) + \sum_{n=1}^{\text{root}} (N_{B_n} - N_{B_{n-1}}) \exp(-h_{B_n}). \tag{2.13}$$

From equations (2.12) and (2.13) it easily follows that for *finite trees* the decay of the autocorrelation function to its equilibrium value is always exponential and dominated by the largest characteristic time,

$$\tau_{\text{root}} = (1/N) \exp(h_{\text{root}}).$$

For infinite trees, on the other hand, either one or both of the following scenarios may take place. (i) Some characteristic times may vanish, indicating that relaxation is unstable in part or all of the tree, which can thus be collapsed to a single state, and (ii) some characteristic times may accumulate to infinity, leading to slower than exponential relaxation at long times. The leading asymptotic behaviour of the autocorrelation function is in this case determined by the asymptotic behaviour of the spectral density

$$\rho(t) = \sum_{\text{branchpoints} B} \frac{S_B - 1}{N} \delta(\tau - \tau_B) \tag{2.14}$$

as $\tau \to \infty$. In the following sections we will analyse this behaviour for trees on which relaxation is everywhere stable.

3. Uniform trees

We begin with the simplest example of an infinite, regular, uniformly multifurcating tree, for which each member of every generation produces b offspring. This is shown in figure 2. Some of the results of this section have already been derived previously (Schreckenberg 1985, Ogielski and Stein 1985, Paladin *et al* 1985, Zheng 1986). If Δh is the height interval between successive generations, the silhouette (equations (2.4*a*, *b*)) is given by

$$s = (1/\Delta h) \log b. \tag{3.1}$$

Using the fact that $N_{B_n} = b^n N_B$, we can easily calculate, from equation (2.13), the inverse characteristic time corresponding to any branch point B of the mth generation, with the result

$$\tau_B^{-1} = [b \exp(-\Delta h)]^m \left(\frac{e^{\Delta h} - 1}{e^{\Delta h} - b} \right)$$

provided $e^{\Delta h} > b$, i.e. $s < 1$. For $s \geq 1$ all characteristic times vanish and relaxation is unstable. Assuming $s < 1$, we finally obtain the autocorrelation function, equation (2.12), in the form

$$\bar{P}(t) = \sum_{m=1}^{\infty} (b-1)b^{-m} \exp\left(-t(b\, e^{-\Delta h})^m \frac{e^{\Delta h} - 1}{e^{\Delta h} - b} \right) \simeq Dt^{-\nu_{\text{uniform}}} \tag{3.2a}$$

where

$$\nu_{\text{uniform}} = s/(1-s) \tag{3.2b}$$

and

$$D = \Gamma\left(\frac{s}{1-s} \right)\left(\frac{e^{\Delta h} - 1}{e^{\Delta h} - b} \right)^{-s/(1-s)} \frac{b-1}{\log b} \frac{s}{1-s} \tag{3.2c}$$

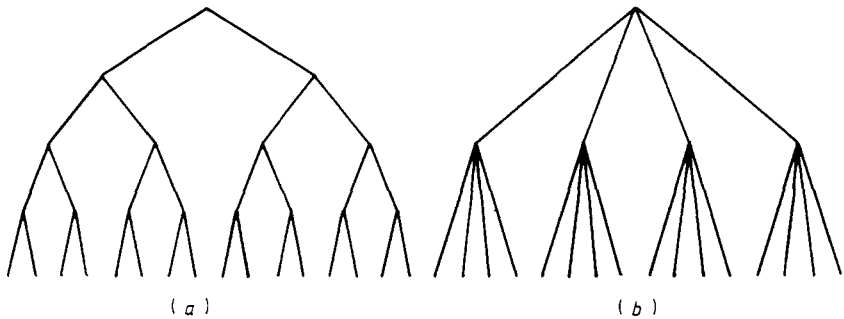with the corrections to the asymptotic behaviour (3.2*a*) falling off exponentially at large t (Erdélyi 1956).

(*a*)   (*b*)

Figure 2. Regular uniformly (*a*) bifurcating and (*b*) tetrafurcating trees; since the (asymptotic) rate of population growth with height is the same, these two trees have the same silhouette, and hence, as discussed in the text, yield the same dynamic critical exponent.

Note that, in contrast to the prefactor D, the dynamical critical exponent ν_{uniform} is only a function of the silhouette and is therefore invariant under $b \to b^{\alpha}$, $\Delta h \to \alpha \Delta h$. For instance, a tree that tetrafurcates at every unit height interval has the same power-law decay as one that bifurcates every half unit interval (see figure 2). One may also relax the condition that the branching ratio and height intervals be generation independent (but still insist that all members of a given generation be indistinguishable). By appropriately bounding $\bar{P}(t)$ from above and below, it is straightforward to show that if the silhouette can be defined and lies between 0 and 1, the dynamic exponent is still given by expression (3.2b). The prefactor D, however, can in this case acquire oscillatory and/or logarithmic time dependence.

By allowing non-constant height intervals among successive generations, one can also study the limiting cases of vanishing or unit silhouette. Indeed, if the height h_m of the mth generation grows faster than linearly with m (so that $s = 0$) the autocorrelation function decays slower than any power of time, implying a $1/f$ frequency spectrum (up to logarithmic corrections). We shall refer to such trees with vanishing silhouette as 'brooms'; we will later show that ultradiffusion may lead to a $1/f$-like spectrum only when the underlying tree is a broom. At the other extreme, the slowest rate of growth of h_m leading to finite characteristic times is given by

$$h_m = m \log b + \alpha \log m \qquad \alpha > 1.$$

It is clear that in this case $s = 1$, and the autcorrelation function is

$$\bar{P}(t) = (b-1) \sum_{m=1}^{\infty} b^{-m} \exp\{-t[m^{-\alpha} + (1 - 1/b)\zeta(\alpha, m+1)]\} \simeq D(t) \exp(-c/t^{\alpha}) \quad (3.3)$$

where $\zeta(\alpha, m+1)$ is the Riemann zeta function, c is a non-universal constant and $D(t)$ decays slower than the leading stretched exponential. The asymptotic behaviour (3.3) is the well known Kohlrausch law.

As opposed to the power-law decay, both the Kohlrausch and the logarithmic relaxation processes seem less generic since they occur at special values of the tree's silhouette.

The expression (3.2b) for the dynamic critical exponent shows that fatter trees relax faster. In order to study the effect of the tree's structure on dynamics, it is therefore reasonable to only compare trees with a fixed silhouette (thus avoiding a heavyweight-featherweight bout). We will, in fact, restrict ourselves to trees with a fixed silhouette slope $0 < s(h) = s < 1$. The following result then shows that *stable* relaxation is fastest on uniform trees.

Theorem 1. (Optimality of infinite uniform trees.) The dynamic critical exponent of any tree with fixed silhouette slope: $0 < s(h) = s < 1$ is bounded from above by $\nu_{\text{uniform}} = s/(1 - s)$, provided there exist some $x < 1$, such that no branch point B has descendants growing faster than $\exp[x(h_B - h)]$ with height.

Remark. This assumption ensures stable relaxation everywhere on the tree. Indeed, if some branch point B has descendants growing like $\exp(h_B - h)$ or faster, then relaxation among these descendants is unstable. The associated diffusion problem should therefore be reformulated on a new tree, obtained by collapsing the subtree generated by B to a single state.

Proof of theorem 1. We first break the expression (2.12) into a sum over generations:

$$\bar{P}(t) = \sum_{m=1}^{\infty} \sum_{B:h_B=m\Delta h} \frac{1}{N} (S_B - 1) \exp(-t/\tau_B).$$

We then use the fact that the average of exponentials is larger than the exponential of the average to obtain

$$\bar{P}(t) \geq \sum_{m=1}^{\infty} \frac{1}{N} A_m \exp(-t/\tau_m) \tag{3.4a}$$

where A_m is the net population increase at the mth generation:

$$A_m = \sum_{B:h_B=m\Delta h} (S_B - 1) = N \exp(-sm\Delta h)(e^{s\Delta h} - 1) \tag{3.4b}$$

while τ_m^{-1} is the average inverse characteristic time of the mth generation:

$$\tau_m^{-1} = \frac{1}{A_m} \sum_{B:h_B=m\Delta h} (S_B - 1)\tau_B^{-1}. \tag{3.4c}$$

Now, for any B belonging to a generation older than m, let $d_m(B)$ stand for the number of descendants of B at the mth generation. The assumption of stable relaxation ensures that $d_m(B) \leq \exp[x(h_B - m\Delta h)]$. Using this and equation (2.13) in (3.4c) one can obtain an upper bound for τ_m^{-1} as follows:

$$\sum_{B:h_B=m\Delta h} (S_B - 1)\tau_B^{-1} = \sum_{B:h_B \geq m\Delta h} N_B \, e^{-h_B}(1 - e^{-\Delta h})(d_{m-1}(B) - d_m(B))$$

$$\leq N(1 - e^{\Delta h}) \sum_{n=0}^{\infty} \exp[-(m+n)\Delta h] \exp[x(n+1)\Delta h]$$

$$\rightarrow \tau_m^{-1} \leq c \exp[m(s-1)\Delta h] \tag{3.5}$$

with

$$c = \frac{(1 - e^{-\Delta h}) \, e^{x\Delta h}}{[\exp(s\Delta h) - 1]\{1 - \exp[(x-1)\Delta h]\}} < \infty.$$

Putting inequality (3.5) back into (3.4a) we finally obtain

$$\bar{P}(t) \geq (e^{s\Delta h} - 1) \sum_{m=1}^{\infty} \exp(-sm\Delta h) \exp\{-tc \exp[m(s-1)\Delta h]\} \simeq \tilde{D}t^{-s/(1-s)}$$

with $\tilde{D}$ a constant, which completes the proof of our theorem.

A possible application of this result is in hierarchical organisations or computer networks in which information signals are transmitted through diffusion. What we have shown is that uniform structures are in this case optimal. Note however that our result concerns only the asymptotic behaviour of the average autocorrelation function, and hence does not in general apply if one is interested in either a particular initial condition or in small finite trees. For the latter we can, nevertheless, prove a similar but weaker optimality result, as follows.

Let $T(M, B)$ be the class of all finite trees with M generations, and an average overall branching ratio b per generation. For a given tree $T \in T(M, b)$ order its characteristic times τ_B in decending order. We say that T_1 relaxes faster than T_2 if for their first non-matching characteristic times $\tau_{B^{(1)}} < \tau_{B^{(2)}}$. Then the following lemma holds.

Lemma 1. (Optimality of finite uniform trees.) Among all trees in $T(M, b)$ for which each member of every generation has at least two sons, the uniform tree relaxes fastest.

Proof. The smallest inverse characteristic time $\tau_{\text{root}}^{-1} = b^M e^{-M\Delta h}$ is the same for all trees in $T(M, b)$. Consider next the inverse characteristic times corresponding to the b members $B^{(1)}, \ldots, B^{(b)}$ of the $(M-1)$th generation:

$$\tau_{B^{(i)}}^{-1} = N_{B^{(i)}} \exp[-(M-1)\Delta h] + (b^M - N_{B^{(i)}}) \exp[-M\Delta h].$$

These have in general different but non-vanishing multiplicities, since by assumption $S_{B^{(i)}} > 1$. Their sum

$$\sum_{i=1}^{b} \tau_{B^{(i)}}^{-1} = b^M \exp[-(M-1)\Delta h] + b^M(b-1) \exp(-M\Delta h)$$

is a constant in $T(M, b)$. Hence, to maximise their minimum, they must all be equal, i.e. $N_{B^{(1)}} = \ldots = N_{B^{(b)}} = b^{M-1}$; the proof of the optimality of uniform trees can now be easily completed by induction.

4. Structural noise

We now turn our attention to uniformly random trees, constructed by allowing the multifurcation number of every branch at every generation to be an independent random variable with probability distribution $P(X)$. These can be considered as uniform trees, with branching ratio

$$\langle X \rangle = \sum_{x=1}^{\infty} P(X)X$$

per generation† and with some structural noise. Our main result is that such noise is irrelevant in that it leads to at most logarithmic modifications of the autocorrelation decay, and thus leaves the dynamic exponent unchanged, i.e.

$$\nu_{\text{random}} = \frac{s}{1-s} \qquad s = \frac{1}{\Delta h} \log\langle X \rangle < 1. \tag{4.1}$$

The reader not interested in the technical details of the demonstration can skip the remainder of this section.

To start with we briefly review some basic facts from the theory of branching processes (Harris 1963). Let X_n be the number of descendants of a given node after n generations, and let $P_n(x)$ be its probability distribution. This clearly satisfies the iteration equation

$$P_n(X) = \sum_{k=1}^{\infty} P(k)\left(\sum_{X_1,\ldots,X_k=1}^{\infty} P_{n-1}(X_1) \ldots P_{n-1}(X_k)\delta_{X,X_1+\ldots+X_k} \right). \tag{4.2}$$

In terms of the generating functionals

$$G_n(s) = \sum_{X=1}^{\infty} P_n(X)s^x$$

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† If $\langle x \rangle$ is not an integer, it can be made into one by raising it to a power and appropriately rescaling the inter-generation gap, as discussed in the previous section.

and since convolution amounts to multiplication, equation (4.2) becomes

$$G_n(s) = g(G_{n-1}(s)) = g^{(n)}(s) \tag{4.3}$$

where $g^{(n)}$ is the nth iterate of the generating functional $g(s)$ of $P(X)$.

Since $g(1) = 1$ for normalisation, we easily find that the average value of X_n is given by

$$\langle X_n \rangle = \frac{\mathrm{d}G_n}{\mathrm{d}s}(1) = \left(\frac{\mathrm{d}g}{\mathrm{d}s}(1) \right)^n = \langle X \rangle^n.$$

Define the random variable $W_n = X_n / \langle X \rangle^n$. The basic result we will need is that, provided the second moment of $P(X)$ exists, W_n converges as $n \to \infty$ with probability one and in mean square to a random variable W with finite variance, and with an absolutely continuous distribution ρ which satifises the stationarity condition that follows from equation (4.2):

$$\rho(W) = \langle X \rangle \sum_{k-1}^{\infty} P(k) \int_0^{\infty} \dots \int_0^{\infty} \mathrm{d}z_1 \dots \mathrm{d}z_{k-1} \, \rho(z_1) \dots \rho(z_{k-1})$$

$$\times \rho(W\langle X \rangle - z_1 \dots - z_{k-1}). \tag{4.4}$$

In words, this means that in a random branching process, the population has the expected exponential growth up to a random prefactor.

To illustrate these facts, consider the simple example of an exponentially decaying distribution given by

$$P(X) = (1 - 1/c)c^X \qquad \text{for } X \in Z^+$$

where $c < 1$ and $\langle X \rangle = 1/(1-c)$. Its generating functional

$$g(s) = \frac{(1-c)s}{1-cs} = \frac{s}{\langle X \rangle - s(\langle X \rangle - 1)}$$

belongs to the class of fractional linear generating functionals, which are closed under iteration. One therefore finds

$$G_n(s) = \frac{s}{\langle X \rangle^n - s(\langle X \rangle^n - 1)} \to P_n(Y) = \frac{(1 - \langle X \rangle^{-n})^Y}{\langle X \rangle^n - 1} \tag{4.5}$$

from which we deduce that W_n converges exponentially fast in n to a random variable with distribution $\rho(W) = \mathrm{e}^{-w}$.

Going back to the problem of diffusion, we would like to average the autocorrelation function over all trees with the appropriate probability distribution. Since averaging over all trees automatically takes care of averaging over initial conditions, it will suffice to consider expression (2.11) for the autocorrelation function, which we rewrite for convenience as

$$P_L(t) \equiv \sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} (N_{L_{n-1}}^{-1} - N_{L_n}^{-1})$$

$$\times \exp \left[-t \left(N_{L_n} \exp(-h_{L_n}) + \sum_{m > n} (N_{L_m} - N_{L_{m-1}}) \exp(-h_{L_m}) \right) \right]. \tag{4.6}$$

According to our previous conventions, the above summation runs over all branch-point ancestors of the initial leaf L. We may, however, also take the summation to run over

all generations, in which case L_n denotes the member of the nth generation on the unique path from L to the root, and $h_{L_n} = n\Delta h$. If L_n is not a branch point (i.e. if it has a unique son) then

$$\Delta N_{L_n} \equiv N_{L_n} - N_{L_{n-1}} = 0$$

and L_n makes a vanishing contribution to the autocorrelation function, as it should.

The asymptotic behaviour of the average autocorrelation function $\bar{P}(t)$ is determined by the large-n behaviour of the average summand $\bar{S}_n$; in order to perform this average, the trick is to note that S_n only depends on the *independent* random variables:

$$W_n \equiv N_{L_n}/\langle X \rangle^n$$

and on all

$$\tilde{W}_m \equiv \Delta N_{L_{m+1}}/\langle X \rangle^m \qquad \text{for } m \geq n$$

whose probability distributions converge at large n to $\rho(w)$ and $P(1)\delta(\tilde{w}) + \bar{\rho}(\tilde{w})$, respectively, where

$$\bar{\rho}(\tilde{w}) = P(2)\rho(\tilde{w}) + P(3)\rho^*\rho(\tilde{w}) + \dots$$

with $*$ denoting convolution of probabilities. This latter distribution follows from the fact that with probability $P(1)$ no new line of descendants other than the one leading to L emanates from L_n, in which case $\tilde{W}_n = 0$, while with probability $P(k+1)$, k new lines do emanate and subsequently grow as (practically infinite) independent random trees. We can thus write for large n:

$$\bar{S}_n \sim \langle X \rangle^{-n+1} \int_{\langle X \rangle^{-n+1}}^{\infty} \frac{\mathrm{d}w}{w} \rho(w) \exp(-tw\langle X \rangle^{n-1} e^{-n\Delta h})$$

$$\times \prod_{m \geq n-1} \left(P(1) + \int_0^{\infty} \mathrm{d}\tilde{w}\, \bar{\rho}(\tilde{w}) \exp\{-t\tilde{w}\langle X \rangle^m \exp[-(m+1)\Delta h]\} \right)$$

$$- \langle X \rangle^{-n} \int_{\langle X \rangle^{-n}}^{\infty} \frac{\mathrm{d}w}{w} \rho(w) \exp[-tw\langle X \rangle^n \exp(-n/\Delta h)]$$

$$\times \prod_{m \geq n} \left(P(1) + \int_0^{\infty} \mathrm{d}\tilde{w}\, \bar{\rho}(\tilde{w}) \exp\{-t\tilde{W}\langle X \rangle^m \exp[-(m+1)\Delta h]\} \right). \qquad (4.7)$$

In the above expression, we kept the vanishingly small cutoff of the w integrations that comes from the requirement that $N_{L_n} = w_n/\langle X \rangle^n \geq 1$; this is not necessary for the w integrations which, as will become clear in a moment, are convergent as $w \to 0$. If this were also true for the w integrations, the summand would obey the homogeneity relation

$$\overline{S_{n+1}}(t) = \langle X \rangle^{-1} \overline{S_n}(t\langle X \rangle e^{-\Delta h})$$

from which we could easily deduce by a change of variables ($z = t\langle X \rangle^n e^{-n\Delta h}$) that

$$\bar{P}(t) = \sum_{n=1}^{\infty} \overline{S_n}(t) \sim t^{-\nu_{\text{random}}} \qquad (4.8)$$

with

$$\nu_{\text{random}} = \frac{\log\langle x \rangle}{\Delta h - \log\langle x \rangle} = \frac{s}{1-s} = \nu_{\text{uniform}}.$$

We will now show that a more careful treatment of the w integration cutoff leads to at most logarithmic modifications of the above power-law decay.

To this end we first prove the following lemma.

Lemma 2. $\rho(w)$ is either bounded, or else diverges at most logarithmically, as $w \to 0$.

Proof. Assume $\rho(w) \sim cw^{-\alpha}$ as $w \to 0$, with $1 > \alpha > 0$ (since ρ must be normalisable). Putting this into the stationarity condition (4.4) one then obtains

$$CW^{-\alpha} = \langle X \rangle P(1) C(W \langle X \rangle)^{-\alpha} (1 + O(W^{1-\alpha}))$$

and hence

$$P(1) = \langle X \rangle^{\alpha-1}.$$

But

$$\langle X \rangle = P(1) + 2P(2) + 3P(3) + \ldots \le P(1) + 2(1 - P(1))$$

so that $\langle X \rangle + \langle X \rangle^{\alpha-1} \le 2$. This cannot be satisfied by any $\langle X \rangle > 1$ which in turn implies that we have derived a contradiction. Thus $\rho(w)$ cannot diverge faster than logarithmically as $w \to 0$.

Note that one can likewise show that $\tilde{P}(\tilde{w})$ diverges at most logarithmically as $\tilde{w} \to 0$, which justifies our throwing away the $\tilde{w}$-integration cutoffs.

Converting the sum over generations to an integral, and changing variables to $z = t \langle X \rangle^n e^{-n\Delta h}$, we can write the average autocorrelation function at large t in the form

$$\bar{P}(t) \sim t^{-\nu_{\text{uniform}}} \left(\int_0^\infty \frac{dz}{z} z^{\nu_{\text{uniform}}} \int_{(z/t)^{\nu_{\text{uniform}}}}^\infty \frac{dw}{w} \rho(w) f(z, w) \right)$$

with $f(z, w)$ a bounded function that decays exponentially at large z. Since $\rho(w)$ is integrable as $w \to \infty$, and has at most a logarithmic singularity as $w \to 0$, we easily conclude that the term in brackets has at most a logarithmic time dependence at large t.

This completes our demonstration that uniformly random trees have the same dynamic exponent as completely ordered uniform trees. One could also verify this assertion directly by calculating the exact average autocorrelation function in the special case of the exponential distributions given by (4.5). Similar results have been obtained by Kumar and Shenoy (1986).

5. Complex hierarchical structures

Both the uniform and the totally random trees are self-similar hierarchical systems, whose parts (or subtrees) are at least on the average identical to the whole. Huberman and Hogg (1986) have argued that they are therefore equally simple structures, which should minimise any physically relevant measure of a tree's complexity. Complexity is in this sense tantamount to lack of self-similarity or to diversity at all levels of the hierarchy. This is to be contrasted with the information-theoretic measure given by Shannon's entropy defined by the size of the smallest algorithm describing how to construct an exact replica of a given tree, and hence maximised by random trees.

Our results place the above ideas in a precise physical context. Indeed, we have demonstrated that for a given tree silhouette, self-similarity leads to a maximum value for the dynamic critical exponent ν, and hence guarantees the fastest relaxation. In this section we will obtain a saturated lower bound for ν, and will in particular show that non-self-similar structures do indeed lead to a slower power-law decay. Thus complexity or structural diversity is reflected in the rate of relaxation, and $-\nu$ is a physically sensible measure of it. Similar results have also been obtained in the context of percolation by Bachas and Wolff (1987), with the word 'complexity' given in that context a precise operational meaning.

Consider a particular unbalanced tree, constructed by allowing the left-half members of every generation to trifurcate, while the right-half members give rise to a single son each, as shown in figure 3(*a*). This is clearly a non-self-similar tree and cannot
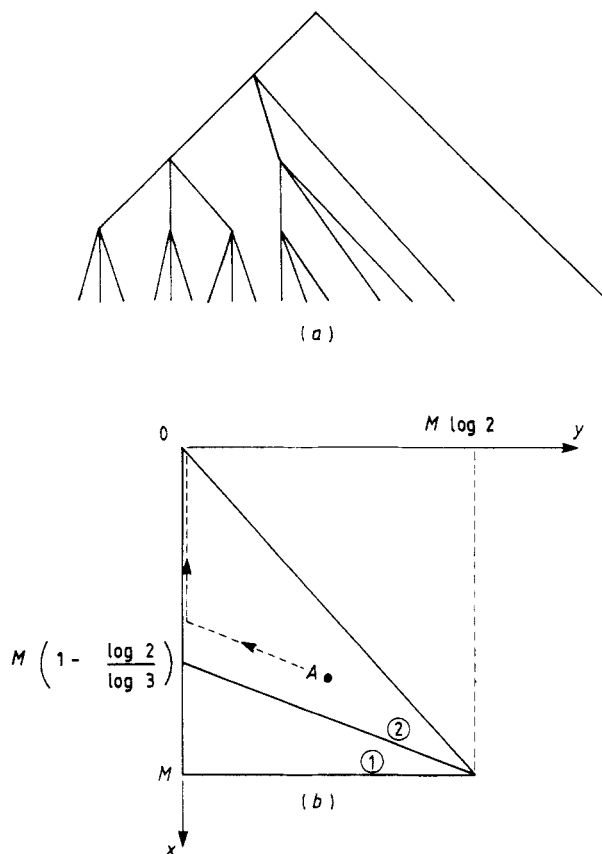

Figure 3. (*a*) An example of a maximally unbalanced tree that saturates the lower bound for the dynamic critical exponent. The left-half members of every generation trifurcate, while the right half give rise to a single son each. (*b*) The orthogonal triangle containing the nodes of this tree, when parametrised as explained in the text. The root lies at the origin. Region 1 is a regularly trifurcating subtree, while region 2 contains those nodes whose descendants trifurcate for a while, but eventually enter the right half of their generation and continue as dead branches thereafter. The broken line is the line of ancestors of a typical node A, along which we integrate to calculate τ_A; its slope is initially $\Delta y/\Delta x = \log 3$, until it hits the left-most branch of the tree, from which point on it continues straight up to the root.

be analysed by renormalisation group techniques. Since the total population doubles at every step, its silhouette is given by

$$s = (1/\Delta h) \log 2.$$

As we will now show, its dynamic critical exponent is $\nu = s$, meaning that relaxation is slower than for the corresponding self-similar structure for which we previously found

$$\nu_{\text{uniform}} = s/(1 - s).$$

Intuitively, this is because different parts of an unbalanced tree relax at different rates, and it is the slowest processes that dominate at long times.

To calculate the average autocorrelation function $\bar{P}(t)$ for our tree of figure 3(a), assume first that the total number of generations is M (M will be taken to infinity in the end). We will label the jth from the left node of the ith from the bottom generation by the coordinates $(x = M - i, y = \log j)$; the tree is then represented by an orthogonal triangle, as shown in figure 3(b), with a hypotenuse slope equal to the silhouette times the inter-generation height interval, $\log 2 = s\Delta h$. Consider now the descendants of a given node (x, y) with $y \neq 0$; these will trifurcate either until they reach the bottom of the tree, or until they enter the right half of their generation, from which point on they will continue as single dead branches until the end; these two cases correspond to regions 1 and 2 respectively in figure 3(b). The number of fertile (i.e. trifurcating) generations following (x, y) is

$$T(x, y) = \begin{cases} M - x & \text{in region 1} \\ \dfrac{(x - 1)\log 2 - y}{\log(3/2)} \pm 1 & \text{in region 2} \end{cases} \tag{5.1}$$

where the ± 1 ambiguity is due to the fact that (at most) one generation of descendants of (x, y) may be partly fertile and partly infertile. We will in the following neglect this ambiguity, since it does not affect the result for the dynamic exponent. Clearly, the number of final descendants or tree leaves generated by node (x, y) is

$$N(x, y) = 3^{T(x,y)}. \tag{5.2a}$$

For $y = 0$ and in region 2, the number of final descendants is actually modified, but we may still express its rate of change with x as

$$\frac{\Delta N}{\Delta x}(x, 0) = 2 \times 3^{T(x,0)}. \tag{5.2b}$$

We are finally in a position to calculate the inverse characteristic time corresponding to the node (x, y), as given by equation (2.13). Converting the sum over ancestors to a line integral, as shown in figure 3(b), and using equations (5.1) and (5.2), we find

$$\tau^{-1}(x, y) = (3 e^{-\Delta h})^{M-x} + \int_{M - M\log 2/\log 3}^{x} \mathrm{d}z (3 e^{-\Delta h})^{M-z}$$

$$+ \int_{0}^{M - M\log 2/\log 3} \mathrm{d}z \exp[-\Delta h(M - z)] 3^{z\log 2/\log(3/2)} \qquad \text{in region 1}$$

$$\tag{5.3}$$

and

$$\tau^{-1}(x, y) = 3^{(x\log 2 - y)/\log(3/2)} \exp[-\Delta h(M - x)]$$

$$+ \int_{x-y/\log 3}^{x} dz \exp[-\Delta h(M - z)] 3^{[z\log 2 - y + (x-z)\log 3]/\log(3/2)}$$

$$+ \int_{0}^{x-y/\log 3} dz \exp[-\Delta h(M - z)] 3^{z\log 2/\log(3/2)} \qquad \text{in region 2} \qquad (5.4)$$

where once again we have dropped various finite multiplicative constants, which cannot modify the leading power law in the asymptotic behaviour of the spectral density, equation (2.14). Note also that the first term on the left-hand side of equation (5.4) should be changed to

$$2^{M} \exp[-\Delta h(M - x)]$$

if $y = 0$.

These expressions simplify considerably if one takes the limit $M \to \infty$ and notes that for almost all nodes (but a set of measure zero) both x and $y \gg \perp$. If

$$s \geq \log 2/\log 3$$

one then finds easily that all inverse characteristic times diverge, implying that relaxation is unstable everywhere on the tree. Let us therefore consider $s \leq \log 2/\log 3$. Up to finite multiplicative constants and exponentially suppressed additive terms we then find

$$\tau^{-1}(x, y) = \begin{cases} (3 e^{-\Delta h})^{M-x} & \text{in region 1} \\ (x \log 2 - y)/3^{\log(3/2)} \exp[-\Delta h(M - x)] & \text{in region 2.} \end{cases}$$

It is now straightforward to analyse the asymptotic behaviour of the average autocorrelation function, equation (2.12), which can be written as

$$P(t) = \int\int_{\text{regions 1 and 2}} dx \, dy \, e^{y} \exp[-t/\tau(x,y)]$$

$$= D_1 t^{-\nu_1} + D_2 t^{-\nu_2} \qquad (5.5)$$

where the two terms in the last expression correspond to the integration over regions 1 and 2 respectively. One finds

$$\nu_1 = \frac{s}{(\log 2/\log 3) - s} \qquad \nu_2 = s. \qquad (5.6)$$

The second of these two exponents, being smaller, dominates at long times and thus describes the asymptotic decay of the average autocorrelation function. We have emphasised the word 'average', because for any given initial condition decay can be much faster; if for example our particle starts out on a tree leaf in the left half of its generation, it can be shown that the autocorrelation function dies out with exponent ν_1, which incidentally is the exponent obtained for a tree that trifurcates uniformly every height interval Δh. Figure 3(b) may actually mislead the reader to think that such 'fast decaying' initial conditions should dominate the average, but this is not the case[†]: the prefactor D_2 does not vanish.

[†] Note that in figure 3(b), y is a logarithmic scale, so that practically the entire population of tree leaves is concentrated in the lower right corner of the triangle.

The dynamic exponent for the unbalanced tree of this section obviously satisfies

$$\nu = s < \nu_{\text{uniform}} = s/(1-s).$$

Thus rearranging the branches of a uniform tree (without changing its silhouette, i.e. adding new branches) can indeed slow down relaxation. Is there a limit to this slowing-down effect or can we go all the way to a logarithmic time decay ($1/f$ noise) by appropriately complexifying the tree's structure? The following result answers this question.

Theorem 2. The dynamic critical exponent of any tree is bounded from below by its silhouette: $\nu \geqslant s$.

Proof. From equation (2.13) it follows easily, since $N_B \geqslant 1$, $N_{B_n} \geqslant N_{B_{n-1}}$, that

$$\tau_B^{-1} \geqslant e^{-h_B}$$

for all tree nodes B. Thus we can bound the average autocorrelation function as follows:

$$\overline{P(t)} \leqslant \sum_{\text{tree nodes } B} \frac{1}{N} (S_B - 1) \exp(-t \, e^{-h_B})$$

$$= \sum_{n=1}^{\infty} (e^{s\Delta h} - 1) \exp(-ns\Delta h) \exp(-t \, e^{-n\Delta h})$$

$$\simeq t^{-s}.$$

This simple theorem thus demonstrates that the unbalanced tree of this section saturates the lower bound for the dynamic exponent, and leads to the slowest allowed relaxation. We should point out that this slowest-relaxing tree is not unique. For instance, any tree for which the bth fraction of each generation b-furcates (with b some integer), while the remaining members continue as dead branches to the end, would give the same dynamic exponent. The prefactor D, as well as the critical silhouette above which relaxation is unstable, do however depend on b.

6. Discussion

Figure 4 summarises our results. The dynamic exponent ν is maximised by both uniform and random trees, and minimised by the unbalanced trees of § 5. The same qualitative behaviour is obtained if instead of ν one plots a measure of the tree's complexity defined by counting the number of non-isomorphic pieces at every generation of the tree, as shown by Huberman and Hogg (1986). More recently, it has been shown that the critical threshold for percolation on a tree, which can be interpreted as the complexity of winning strategies in a game, is also minimised by uniform and random trees and maximised by the trees of § 5 (Bachas and Wolff 1987). We may therefore say that the rate of relaxation is sensitive to the physical complexity or lack of self-similarity, rather than to the structural noise or Shannon entropy of the underlying tree. It would be interesting to see whether quantitative relations among these various complexity measures can be obtained.

Besides slowing down relaxation, the complexity or absence of self-similarity has another more spectacular effect, when one considers thermally activated processes. Assuming in this case that the hopping rates are given as
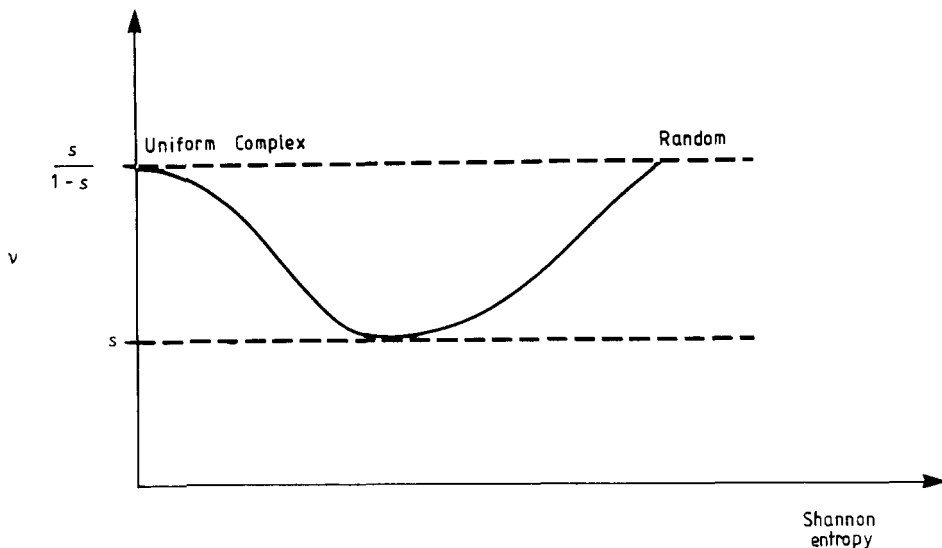
$$\varepsilon_{ij} = \exp(-\Delta V_{ij}/T) \tag{6.1}$$

Figure 4. A schematic plot of the dynamic exponent ν against the Shannon entropy, or detailed information content of the underlying tree. Here s is the silhouette of the tree, which is held constant, and the broken lines are rigorous bounds.

with ΔV_{ij} an energy barrier, we easily conclude from the definition (2.4) that the silhouette is proportional to the temperature. Therefore the temperature dependence of the dynamic exponent is given by

$$\nu_{\text{leastcomplex}}(T) = \begin{cases} T/(T_c - T) & T < T_c \\ \infty & T > T_c \end{cases} \tag{6.2a}$$

for self-similar (uniform or random) trees and

$$\nu_{\text{mostcomplex}}(T) = \begin{cases} cT & T < T_c' \\ \infty & T > T_c' \end{cases} \tag{6.2b}$$

for the most complex trees of § 5. This is illustrated in figure 5. Note in particular that the transition to instability is continuous in the former case and discontinuous in the latter.
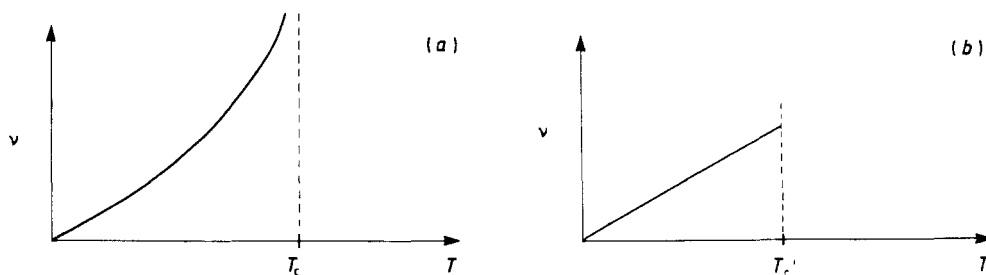

Figure 5. The temperature dependence of the dynamic exponent, assuming hopping rates are given by equation (6.1), for (a) self-similar and (b) maximally complex trees. Note that the transition is continuous in the former case and discontinuous in the latter.

Before proceeding we should point out that ultradiffusion is not just a naive model of the relaxational dynamics of systems with many timescales. It can be actually considered as a universal description of such dynamics, in two different ways.

(i) In real space one may describe the spreading of an excitation by lumping together in larger and larger blocks the degrees of freedom that relax at characteristic times $\tau_1 < \tau_2 < \ldots$, as has been suggested by Palmer *et al* (1984). Note that for systems with disorder the ensuing hierarchical tree is in general non-uniform.

(ii) In configuration space one may describe relaxation as a stochastic motion of an ensemble of particles in a valley landscape of metastable states (Dotsenko 1985). At sufficiently low temperature, the hopping rate between two such states is

$$\varepsilon_{ij} \simeq \exp(-\text{minmax}(ij)/T)$$

where $\text{minmax}(i,j)$ is the minimum over all paths from i to j of the maximum energy barrier encountered along the path. This satisfies the ultrametric property

$$\text{minmax}(i,j) \leqslant \max\{\text{minmax}(i,k);\ \text{minmax}(j,k)\}$$

since one can always go from i to j via k, and hence defines a hierarchical structure in the space of metastable states. We owe this argument to Mézard (1986).

It is this second description of relaxation as ultradiffusion in the space of metastable states that is relevant for studying the temperature dependence of the dynamic exponent. The simple assumption (6.1), however, is *a priori* justified only in the limit of vanishing temperature. As T is raised, it is in general invalidated for several reasons. To begin with, entropy comes into play and hopping rates are not simply determined by a maximum energy barrier, so that ultrametricity of the transition matrix may be destroyed. One may simply hope that the ultrametricity of equilibrium states, demonstrated in the mean-field spin glass (Sherrington and Kirkpatrick 1975, Mézard *et al* 1984a, b) and conjectured in a variety of other systems (Palmer 1986), implies also an exact or approximate hierarchy of hopping rates among long-lived metastable states.

Even if this is so, free-energy barriers would in general be T dependent, so that the silhouette is not simply proportional to T. Equations (6.2a) and (6.2b) should therefore be replaced by

$$\nu_{\text{self-similar}}(T) = \begin{cases} s(T)/(1 - s(T)) & T < T_c \\ \infty & T > T_c \end{cases} \qquad (6.3a)$$

and

$$\nu_{\text{mostcomplex}}(T) = \begin{cases} s(T) & T < T'_c \\ \infty & T > T'_c \end{cases} \qquad (6.3b)$$

where $s(T_c) = 1$ in the first case, and $s(T'_c) = \log 2/\log 3$ for the trees of § 5. The precise form of $\nu(T)$ now depends on the unknown function $S(T)$. If, however, the silhouette changes continuously with temperature, the nature of the transition is robust: it is continuous for self-similar trees and discontinuous for complex trees. Note incidentally that the transition described in equation (6.2) is unphysical, since relaxation became unstable above the critical temperature. This is, however, only due to the long-range nature of hoppings and can be easily rectified: by introducing an exponential T-independent cutoff that respects the ultrametric structure of the space, we can ensure that the silhouette is bounded from above at all temperatures, and the transition is to a region of exponential relaxation.

It is tempting to compare these predictions of ultradiffusion with what is known for the mean-field spin glass: as shown by Sompolinsky and Zippelius (1981, 1982) relaxation in the latter is exponential above the glassy transition and changes over to a power law with exponent

$$\nu(T) = \frac{1}{2} - \frac{T_c - T}{\pi T_c} + O(T_c - T)^2$$

below. Does this imply that the hierarchical tree in the Sherrington–Kirkpatrick model (Sherrington and Kirkpatrick 1975, Mézard *et al* 1984) is complex?

More work is necessary to answer such questions and to unravel the rich properties of systems that defy the simplicity of scaling laws.

Acknowledgment

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